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Embedding digraphs of small size

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Abstract

Let $D=(V,A)$ be a digraph without loops and multiple arcs of order $n \geqslant 2$. We say that D is *embeddable* if there is a permutation $\phi : V \to V$ (called *complementing permutation*) such that $(\phi(x), \phi(y)) \notin A$ for every arc $(x,y) \in A$. In this paper we prove that *if* $|A| < \lfloor 3(n-2)/2 \rfloor$ *then* D *is embeddable* and, moreover, there is a complementing permutation which is a composition of disjoint transpositions when n is even or a composition of disjoint transpositions and a fixed point when n is odd.

We consider only finite digraphs and graphs without loops and multiple arcs and edges. Our terminology and notation are standard unless otherwise stated. The size of a digraph D will be denoted by $e(D)$. A digraph $D=(V,A)$ (graph $G=(V,E)$) is said to be *embeddable in its complement*, briefly: *embeddable*, if there is a permutation ϕ : $V \to V$ such that $(\phi(x), \phi(y)) \notin A$ ($xy \notin E$) for every arc $(x,y) \in A$ (edge $xy \in E$). The permutation ϕ is then called a *complementing permutation* of D (G, respectively). A digraph D is said to be *self-complementary* if it is isomorphic to its complement $\bar{D}$, i.e. when there is a complementing permutation which is an isomorphism of the digraphs D and $\bar{D}$.

The problem of finding the maximum number $f(n)$ such that every undirected graph of order n and size at most $f(n)$ is embeddable has been independently solved by Bollobás and Eldridge [2], Burns and Schuster [3], Ganter et al. [6] and Sauer and Spencer [7] who proved the following:

Theorem 1. *Every graph G of order n and size at most $n-2$ is embeddable.*

In fact, the authors of [2,6,7] proved stronger results implying Theorem 1. Moreover, all graphs of order n and size $n-1$ which are not embeddable have been given in [4,6]. Faudree et al. [5] characterized all non-embeddable graphs of order and size n.

Much less is known about digraphs. In [1] the following has been proved:

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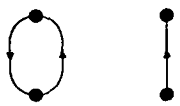

Fig. 1. The exceptional digraph for Theorem 3.

Theorem 2. *Every digraph of order $n \geqslant 3$ and size at most n is contained in a self-complementary digraph.*

It is evident that a digraph contained in a self-complementary digraph is embeddable. Theorem 2 is far from the expectation formulated in [1].

Conjecture 1. Every digraph of order n and size at most $2n-3$ is contained in a self-complementary digraph of order n unless n is even and D is isomorphic to the digraph D' or its converse, where D' is defined by $V(D') = \{x_1, \dots, x_n\}$ and $E(D') = \{x_1x_2, x_1x_3, \dots, x_1x_{n-2}, (x_1, x_{n-1}), (x_1, x_n), (x_{n-1}, x_n)\}$.

Lemma 1 (Benhocine and Wojda [1]). *Let $D = (V, A)$ be a digraph of order n. Then D is contained in a self-complementary digraph of order n if and only if there exists a permutation ϕ of V whose cycles are even (except one ot length one if n is odd) such that $(\phi^{2m+1}(x), \phi^{2m+1}(y)) \notin A$ for every $(x, y) \in A$ and for every integer m.*

We announce the following:

Theorem 3. *Every digraph $D = (V, A)$ of order $n \geqslant 2$ and size at most $3(n-2)/2$ is embeddable. Moreover, when D is not isomorphic to the vertex disjoint union of a 2-cycle and an arc (see Fig. 1) then there is a complementing permutation ϕ of D whose cycles are disjoint transpositions, except one which is a fixed point when n is odd.*

Observe that, by Lemma 1, every digraph D of order $n \geqslant 2$ and size at most $3(n-2)/2$ is contained in a self-complementary digraph.

Our purpose is to prove Theorem 4 which is a slightly weaker version of Theorem 3. In this way we avoid studying a big number of cases when we obtain the exceptional digraph by removing some arcs from a considered digraph D.

Theorem 4. *Every digraph $D = (V, A)$ of order $n \geqslant 2$ and size less than $3(n-2)/2$ is embeddable and there is a complementing permutation of D which is a composition of disjoint transpositions when n is even or a composition of disjoint transpositions and a fixed point when n is odd.*

Proof. The arc with beginning vertex a and end vertex b will be denoted by (a, b). By $<a, b>$ we denote an arc (a, b) or (b, a). We say that arcs $<a, b>$ and $<c, d>$

have the same orientation if either $(a,b) \in A$ and $(c,d) \in A$ or $(b,a) \in A$ and $(d,c) \in A$. Otherwise we say that $<a,b>$ and $<c,d>$ *have the opposite orientation.* For brevity we call *t-permutation of D* the complementing permutation of D which is a composition of disjoint transpositions except one which is a fixed point when n is odd.

The proof is by induction on n. If $W \subset V$, then $D - W$ is the subdigraph of D induced by $V - W$. We distinguish several cases and subcases in each of which we find a subset W of vertices of D such that the graph $D'=D-W$ verifies the assumptions of Theorem 4, i.e. D' has a *t-permutation* σ'. We extend σ' to a *t-permutation* σ of D.

The reader may easily check that Theorem 4 is true when $2 \leqslant n \leqslant 5$. So let us suppose that $n \geqslant 6$, D is the digraph of order n and size less than $\frac{3}{2}(n-2)$ and the theorem is true for the digraphs with order less than n.

Without loss of generality we may assume that $e(D) = \lfloor [3(n-2)-1]/2 \rfloor$.
We often use the following formulas:

$$\left\lfloor \frac{3(n-2)}{2} \right\rfloor - \left\lfloor \frac{3(n-3)}{2} \right\rfloor = \begin{cases} 2 & \text{for } n \text{ even,} \\ 1 & \text{for } n \text{ odd,} \end{cases} \tag{1}$$

$$\left\lfloor \frac{3(n-2)}{2} \right\rfloor - \left\lfloor \frac{3(n-4)}{2} \right\rfloor = 3. \tag{2}$$

Lemma 2. *If there is an isolated vertex (i.e. a vertex of degree 0) in D then there is a t-permutation of D.*

Proof. Let x_0 be an isolated vertex of D. The reader may easily check that there is a vertex x_1 in D such that $d(x_1,D) \geqslant 3$. Then $D' = D - \{x_0,x_1\}$, $\sigma = (x_0,x_1)\sigma'$. □

So, from now on, we shall assume that $\delta(D) \geqslant 1$.

Lemma 3. *If there is a vertex x_0 of D such that $d(x_0,D) = 1$ then there is a t-permutation of D.*

Proof. We assume that $d^+(x_0,D)=1$ and $d^-(x_0,D)=0$ (the case when $d^-(x_0,D)=1$ is similar). If n is odd then, by the induction hypothesis, there is a t-permutation σ' of $D' = D - \{x_0\}$ and, clearly, $\sigma = (x_0)\sigma'$ is a t-permutation of D. So let us suppose that n is even. Let x_1 be the neighbour of x_0. We consider three cases.

Case 1: $d(x_1,D) \geqslant 3$.

Then $D' = D - \{x_0,x_1\}$ and $\sigma = (x_0,x_1)\sigma'$.

Case 2: $d(x_1,D) = 2$.

Denote by x_2 the second neighbour of x_1. We distinguish four subcases.

Subcase 2.1: $d(x_2,D) \geqslant 4$.

If x_2 is the neighbour of all the vertices of $D-\{x_0, x_1, x_2\}$ then one of them has to be of degree 1 we have Case 1. Otherwise there is a vertex $x_3 \in \{x_0, x_1, x_2\}$ nonadjacent

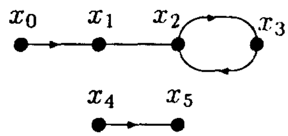

Fig. 2. $\sigma = (x_1,x_2)(x_0,x_4)(x_3,x_5)$.

with x_2. Then $e(D - \{x_0,x_1, x_2, x_3\}) < 3((n-4)-2)/2$ and for any t-permutation σ' of $D' = D - \{x_0, x_1, x_2, x_3\}$ $\sigma = (x_0, x_3)(x_1, x_2)\sigma'$ is a t-permutation of D.

Subcase 2.2: $d(x_2,D) = 3$.

The reader may check that if all neighbours of x_2 have their degree less than or equal to 2 and all the vertices nonadjacent with x_2 have degrees at most 1, then $n \leqslant 6$. For $n = 6$ we obtain either Case 1, or D is a digraph from Fig. 2.

If there is a vertex x_3 in $D - \{x_0,x_1,x_2\}$ which is not adjacent to x_2 and $d(x_3,D) \geqslant 2$ then we repeat the arguments of the Subcase 2.1.

So we may suppose that all vertices nonadjacent with x_2 have degree 1. If there is a vertex x_3 in $D - \{x_0,x_1,x_2\}$ adjacent to x_2 by a symmetric edge then x_3 has degree at least 3 and is the neighbour of a pendent vertex. We apply Case 1.

If the arc joining x_2 with x_3 is antisymmetric, where x_3 is a vertex of degree at least 3, then one of the permutations, $(x_1, x_2)(x_0, x_3)\sigma'$, $(x_0, x_2)(x_1, x_3)\sigma'$ extends σ' to a t-permutation of D for every t-permutation σ' of $D - \{x_0, x_1, x_2, x_3\}$.

Subcase 2.3: $d(x_2,D) = 2$.

Let x_3 be the neighbour of x_2 in $D - \{x_0,x_1,x_2\}$. Suppose first that either $d(y,D) \geqslant 4$, $y = x_3$, or there is a vertex $y \in V(D) - \{x_1,x_3\}$ such that $d(y,D) \geqslant 3$. Then $e(D - \{x_0,x_1,x_2,y\}) < 3(n-6)/2$ and by the induction hypothesis there is a required t-permutation σ' of $D - \{x_0,x_1,x_2,y\}$. The reader may check that then one of the permutations $\sigma = (x_1,x_2)(x_0,y)\sigma'$ or $\sigma = (x_0,x_2)(x_1,y)\sigma'$ is a t-permutation of D. Assume that $d(x_3,D) \leqslant 3$ and $d(y,D) \leqslant 2$ for any y which is not adjacent with x_2. Then,

$$\frac{3(n-2)}{2} - 1 = e(D) \leqslant \frac{1}{2}\left(\sum_{n=0}^{3} d(x_n) + 2(n-4)\right) = n$$

and therefore $n \in \{6,8\}$. Let $n = 8$. Because of degrees of vertices of $D - \{x_0,x_1,x_2,x_3\}$ we have $d(x_3,D) = 3$. Let us denote by H the complement of the underlying graph of $D' = D - \{x_0,x_1,x_2,x_3\}$. Let us suppose that H has a hamiltonian path $P = (y_0, y_1, y_2, y_3)$. We may assume that y_3 is not joined with x_3 by any symmetric arc. Then $\sigma = (x_0,y_0)(x_1,y_1)(x_2,y_2)(x_3,y_3)$ is a required t-permutation of D. So let us suppose that there is no hamiltonian path in H. Observe that then H is either a star $K_{1,3}$ or the union of a triangle and an isolated vertex, $K_3 \cup K_1$. Then D' is either the union of an oriented triangle and an isolated vertex or an oriented star. The second case is impossible because then the center of the star has degree at least 3. If D' is the union of a triangle and an isolated vertex y then x_3 has to be joined with y by a symmetric arc. The reader will easily find a t-permutation of D. The case $n = 6$ is left for the reader.

Subcase 2.4: $d(x_2,D) = 1$.

If there is a vertex x_3 such that $d(x_3,D) \geqslant 4$, then we may find a t-permutation σ' of $D - \{x_0,x_1,x_2,x_3\}$ and $\sigma = (x_0,x_1)(x_2,x_3)\sigma'$ is a t-permutation of D. The reader may easily check that if there is a vertex of degree 1 then we have one of the cases considered above. It is also easy to see that otherwise $D - \{x_0,x_1,x_2\}$ has $n-6$ vertices of degree 3 and three vertices of degree 2. If $n \geqslant 12$ then there are two nonadjacent vertices of degree 3 in D. Moreover, if $n = 10$ and any two vertices of degree 3 are adjacent then $D - \{x_0,x_1,x_2\}$ is the vertex-disjoint union of a tournament T_4 and a triangle. So for $n \geqslant 10$ we may choose three vertices a,b and c of $D - \{x_0,x_1,x_2\}$ such that $d(a,D) = d(b,D) = 3$ and c is nonadjacent with $\{a,b\}$. Then $e(D - \{x_0,x_1,x_2,a,b,c\}) \leqslant [3(n-8)/2] - 1$ and therefore, by induction hypothesis, there is a t-permutation σ' of $D - \{x_0,x_1,x_2,a,b,c\}$. Hence $\sigma = (x_0,a)(x_1,c)(x_2,b)\sigma'$ is a required t-permutation of D. So let us suppose that $n = 8$. Then there are vertices x_3 and x_4 in $D - \{x_0,x_1,x_2\}$ such that $d(x_3,D) = 3$, x_3 and x_4 are not adjacent. Let a,b and c be the remaining vertices of $D' = D - \{x_0, \ldots, x_4\}$. Since $e(D') \leqslant 1$ we may suppose that c is not adjacent with a and b. Then $\sigma = (x_0,a)(x_1,c)(x_2,b)(x_3,x_4)$ is a t-permutation of D. If $n=6$ then $D - \{x_0,x_1,x_2\}$ is a triangle and $\sigma = (x_1,a)(x_0,x_2)(b,c)$ is a required permutation, where $V(D - \{x_0,x_1,x_2\}) = \{a,b,c\}$.

Case 3: $d(x_1) = 1$.

If $n=6$ then D is the disjoint union of a cycle $C_4 = (y_1,y_2,y_3,y_4,y_1)$ and an isolated arc (x_0,x_1). In this case $\sigma = (x_0,y_2)(x_1,y_4)(y_1,y_3)$ is a t-permutation of D.

If $n \geqslant 6$ then there is a vertex x_2 such that $d(x_2,D) \geqslant 3$.

Let us suppose that $d(x_2,D) \geqslant 5$. Then there is a vertex x_3 in $D - \{x_0,x_1,x_2\}$ such that x_3 is not joined to x_2 by a symmetric arc. There is a t-permutation σ' of $D' = D - \{x_0,x_1,x_2,x_3\}$. Hence one of the permutations: $\sigma_1 = (x_0,x_2)(x_1,x_3)\sigma'$ or $\sigma_2 = (x_0,x_3)(x_1,x_2)\sigma'$ is a required t-permutation of D.

Let us suppose now that $d(x_2,D) = 4$. Then for $n \geqslant 8$ there is a vertex x_3 in $D - \{x_0,x_1\}$ not adjacent with x_2. Set σ' to be a t-permutation of $D' = D - \{x_0,x_1,x_2,x_3\}$. Then both permutations σ_1 and σ_2 are t-permutations of D.

Finally, we assume that there are no vertices of degree greater than 3. For $n \geqslant 10$ there are at least three vertices x_2, x_3, x_4 of degree 3. We may suppose that x_2 and x_3 are not joined by a symmetric arc. Let σ' be a t-permutation of $D' = D - \{x_0,x_1,x_2,x_3\}$. Then σ_1 or σ_2 is a t-permutation of D.

When $n = 8$ there are at least two vertices of degree 3. If there are two vertices x_2 and x_3 of degree 3 such that they are not joined by a symmetric arc then the rest of the proof runs as before. Otherwise, there are exactly two vertices of degree 3 and four vertices of degree 2 in D. We choose one vertex x_2 of degree 3 and one vertex x_3 of degree 2 such that, x_2 is not adjacent to x_3. Then σ_1 and σ_2 are t-permutations of D. $\square$

Lemma 4. *If $\delta(D) = 2$ then there is a t-permutation of D.*

Proof. The proof of Lemma 4 is divided into a sequence of claims. The first one is evident.

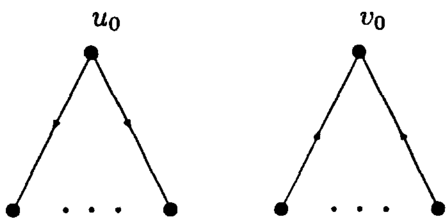

Fig. 3.

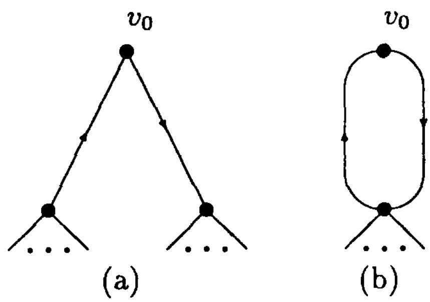

Fig. 4.

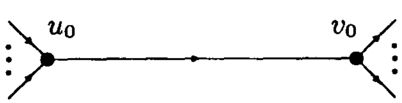

Fig. 5.

Claim 1. *If v_0 and u_0 are such vertices of D that $d^-(v_0,D)=0$ and $d^+(u_0,D)=0$ then there is a t-permutation of D, see Figs.* 3 *and* 4.

Claim 2. *If n is odd and there is a vertex v_0 in D such that $d^+(v_0,D)=d^-(v_0,D)=1$ then there is a t-permutation of D.*

Proof. The digraph $D-\{v_0\}$ verifies the induction hypothesis, hence it is embeddable. Let σ' be a t-permutation of $D-\{v_0\}$. Since $n-1$ is even, σ' has no fixed point and $(v_0)\sigma'$ is a t-permutation of D. □

Claim 3. *If u_0 and v_0 are such vertices of D that $(u_0,v_0)\in A$, $(v_0,u_0)\notin A$ and $d^+(u_0,D)=d^-(v_0,D)=1$. Then there is a t-permutation of D, see Fig.* 5.

Claim 4. *If there are two adjacent vertices v_0 and v_1 such that $d(v_0,D)=d(v_1,D)=2$ then there is a t-permutation of D, see Fig.* 6.

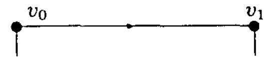

Fig. 6. $d(v_0,D) = d(v_1,D) = 2$.

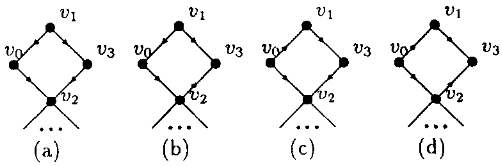

Fig. 7.

Proof. If v_0 and v_1 are joined by a symmetric arc then we consider two nonadjacent vertices x_0 and x_1 of $D - \{v_0, v_1\}$. Let σ' be a t-permutation of $D' = D - \{v_0, v_1, x_0, x_1\}$. Then $(v_0, x_0)(v_1, x_1)\sigma'$ is a t-permutation of D.

From now on, we assume that the arc joining v_0 to v_1 is not symmetric. Suppose that v_0 and v_1 have a common neighbour u. By the induction hypothesis, $D - \{v_0, v_1\}$ has a t-permutation σ'. Then $(v_0, v_1)\sigma'$ is a t-permutation of D unless n is odd, u is a fixed point of σ' and the arcs $<v_0, u>$, $<v_1, u>$ have the same orientation. Then one of the vertices v_0, v_1 verifies the assumption of the Claim 3.

So, we may assume that v_0 is adjacent with a vertex v_2, $v_2 \neq v_1$, v_1 is adjacent with v_3, $v_3 \neq v_0$, $v_3 \neq v_2$. Then, by Claims 1 and 3 arcs $<v_0, v_2>$, $<v_1, v_3>$ have the same orientation. Without loss of generality we may assume that $(v_0, v_2) \in A$ and $(v_1, v_3) \in A$. If v_2 and v_3 are joined by a symmetric arc then (v_2, v_3) is not one of the transpositions in the decomposition of any t-permutation of $D' = D - \{v_1, v_2\}$ σ' into vertex disjoint cycles and therefore $\sigma = (v_0, v_1)\sigma'$ is a required t-permutation. So we may suppose that there is no symmetric arc joining v_2 with v_3. The reader may easily check that each t-permutation of the digraph $D - \{v_0, v_1, v_2, v_3\}$ may be extended to a t-permutation of D by adding two vertex disjoint transpositions on the set $\{v_0, v_1, v_2, v_3\}$. So we may suppose that the vertices v_0, v_1, v_2 and v_3 are incident with at most five arcs. Without loss of generality we may assume that $d(v_3, D)$=2. Let us suppose, first, that the vertices v_2 and v_3 are adjacent. A subdigraph induced by $\{v_0, v_1, v_2, v_3\}$ is depicted in Fig. 7.

In the case pointed out in Fig. 3(a) we apply the induction hypothesis to the digraph $D - \{v_0, v_3\}$ and then we extend a t-permutation σ' of $D - \{v_0, v_3\}$ to $\sigma = (v_0, v_3)\sigma'$. In the case pointed out in Fig. 3(c) we apply Claim 3 and in the remaining cases we apply Claim 1.

So we may assume that v_2 and v_3 are not adjacent and $d(v_2, D)$= $d(v_3, D) = 2$.

Without loss of generality we may suppose that $(v_1, v_0) \in A$. Applying Claim 3 we deduce that $d^+(v_2, D) = 0$. Then using Claim 1 ends the proof of Claim 4. $\square$

Claim 5. *If there are six vertices $v_1, \ldots, v_6$ of D such that $|N(v_i, D)|$=2 and $N(v_i, D) \cap N(v_j, D) = \emptyset$ for $1 \leqslant i < j \leqslant 6$ then, there is a t-permutation of D.*

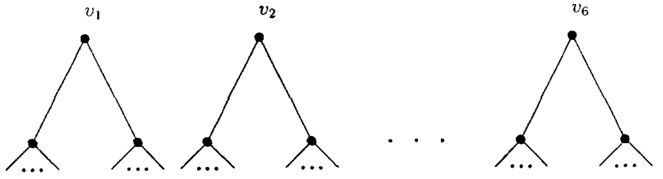

Fig. 8. $d(v_i,D) = 2$, $|N(v_i,D)| = 2 (i = 1,\ldots,6)$.

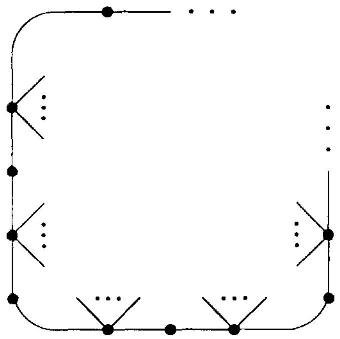

Fig. 9.

Proof. To prove Claim 5, it is sufficient to consider a t-permutation σ' of $D' = D - \{v_1, \ldots, v_6\}$ and extend it to a t-permutation of D by adding three disjoint transpositions on the set $\{v_1, \ldots, v_6\}$. Observe that it may be supposed that σ' does not contain any transpositions of two neighbours of the same vertex v_i $(i=1, \ldots, 6)$ (Fig. 8). We leave the details to the reader. □

Claim 6. *If D contains an even cycle $C = (u_1, v_1, u_2, v_2, \ldots, u_k, v_k, u_1)$ such that $d(v_i,D) = 2$ for $i = 1, \ldots, k$ then there is a t-permutation of D.*

Proof. Suppose first that the number $i(C)$ of arcs incident to C is at least $3k$ (Fig. 9). Then $D' = D - V(C)$ has a t-permutation σ' and $\sigma = (u_1,v_1)\cdots(u_k,v_k)\sigma'$ is a t-permutation of D. If $i(C) < 3k$ then there are i_0 and j_0, $i_0 \neq j_0$, such that u_{i_0} and u_{j_0} are adjacent and $\min\{d(u_{i_0},D), d(u_{j_0},D)\} = 3$. Without loss of generality we may suppose that $i_0 < j_0$ and $j_0 - i_0$ is as small as possible. We consider two cases.

Case 1: $k = 2$.

There are two vertices a and b in $D - V(C)$ that are not adjacent. Let σ' be a t-permutation of $D' = D - \{u_1, v_1, u_2, v_2, a, b\}$. Then $\sigma = (u_1,a)(u_2,b)(v_1,v_2)\sigma'$ is a t-permutation of D.

Case 2: $k > 2$.

Without loss of generality we may suppose that $i_0 = 1$ and $d(u_{j_0},D) = 3$. Then $e(D - \{u_1, u_2, \ldots, u_{j_0}, v_1, v_2, \ldots, v_{j_0}\}) < 3((n - 2j_0) - 2)/2$ and there is a t-permutation σ' of $D - \{u_1, \ldots, u_{j_0}, v_1, \ldots, v_{j_0}\}$. Hence $\sigma = (u_1, v_1)(u_2, v_2)\ldots(u_{j_0}, v_{j_0})\sigma'$ is a t-permutation of D. □

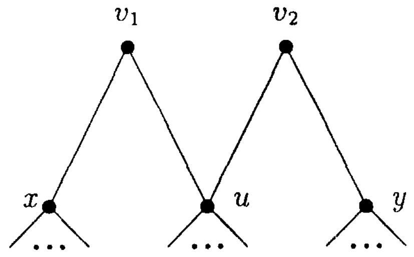

Fig. 10.

Claim 7. *There are three vertices* v_1, v_2 *and* u (*Fig.* 10.) *such that*

(1) $d(v_1,D)=d(v_2,D)=2$;

(2) $|N(v_1,D)|=|N(v_2,D)|=2$;

(3) $N(v_1,D)\cap N(v_2,D)=\{u\}$.

Then either there is a t-permutation of D *or* $d(u,D)\geqslant 4$.

Proof. Observe that by Claim 4 we may assume that v_1 and v_2 are nonadjacent and any vertex of $N(v_1,D)\cup N(v_2,D)$ has the degree at least 3. Let us denote $N(v_1,D)=\{u,x\}$ and $N(v_2,D)=\{u,y\}$. By Claim 6 we may assume that $x\neq y$. Suppose that $d(u,D)=3$. Let s be the third neighbour of u. By Claims 1–3 it is sufficient to consider seven cases. In each of them we find a t-permutation σ of D.

Case 1: $x=s$.

In the remaining cases we assume that $s\neq x$ and $s\neq y$.

Case 2: $(s,y)\in A$. and $(y,s)\in A$.

In cases 3–7 we assume that vertices s and x are not joined by a symmetric arc and vertices s and y are not joined by a symmetric arc.

Case 3: $<v_1,x>$, $<u,s>$, $<v_1,u>$, $<v_2,u>$ and $<y,v_2>$ have the same orientation.

Case 4: $<v_1,u>$, $<v_2,u>$, $<u,s>$, $<x,v_1>$ and $<y,v_2>$ have the same orientation.

Case 5: n is even and $<v_1,x>$, $<u,s>$, $<v_2,y>$, $<u,v_1>$ and $<v_2,u>$ have the same orientation.

Case 6: $<v_1,x>$, $<u,s>$, $<v_2,y>$, $<v_1,u>$ and $<v_2,u>$ have the same orientation.

Case 7: n is even and $<x,v_1>$, $<s,u>$, $<y,v_2>$, $<v_1,u>$ and $<v_2,u>$ have the same orientation.

Set $D_1'=D-\{v_1,v_2,u,x\}$, $D_2'=D-\{v_1,v_2,u,y\}$, $D_3'=D-\{v_1,v_2,u,s\}$ and $D_4'=D-\{v_1,v_2\}$. Denote by σ_i' a t-permutation of D_i', $i=1,2,3,4$. In Cases 1–4, $\sigma=(x,v_1)(u,v_2)\sigma_1'$ is a t-permutation of D. In Case 5 if x is not adjacent with s or arcs $<x,s>$ and $<v_1,u>$ have the same orientation then $\sigma=(u,v_1)(x,v_2)\sigma_1'$ is a t-permutation of D. If arcs $<x,s>$ and $<v_1,u>$ have the opposite orientation, then $\sigma=(v_1,v_2)(u,s)\sigma_3'$ or $\sigma=(v_1,u)(v_2,s)\sigma_3'$ is a t-permutation of D. In Case 6 if x

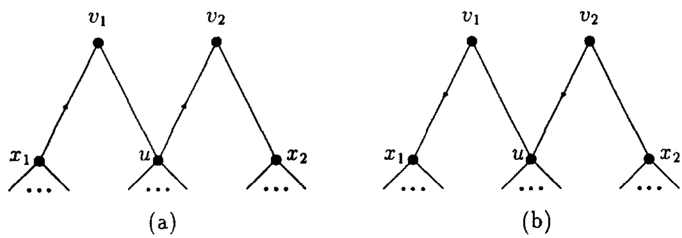

Fig. 11. x_1 and u are nonadjacent.

is not adjacent with s or arcs $<x,s>$ and $<v_1,u>$ have the opposite orientation then $\sigma=(u,v_1)(x,v_2)\sigma_1'$ is a t-permutation of D. If y is not adjacent with s or arcs $<y,s>$ and $<v_2,u>$ have the opposite orientation then $\sigma=(u,v_2)(y,v_1)\sigma_2'$. Otherwise $\sigma=(v_1,v_2)(u,s)\sigma_3'$ or $\sigma=(v_1,u)(v_2,s)\sigma_3'$ is a t-permutation of D. Finally in Case 7 if x and y are joined by a symmetric arc then $\sigma=(v_1,v_2)\sigma_4'$ is a t-permutation of D. Otherwise $\sigma=(v_1,v_2)(x,u)\sigma_1'$ or $\sigma=(v_1,v_2)(y,u)\sigma_2'$ is a t-permutation of D. □

Claim 8. *Let v_1 and v_2 be two vertices of degree 2 such that*
(1) $|N(v_1,D)|=|N(v_2,D)|=2$,
(2) $|N(v_1,D)\cap N(v_2,D)|=1$,
and let u, x_1 and x_2 be such vertices (Fig. 11.) that $N(v_1,D)=\{u,x_1\}$ and $N(v_2,D)=\{u,x_2\}$. If $N(u,D)\cap N(x_1,D)=\{v_1\}$ and the arcs $<v_1,x_1>$ and $<v_2,u>$ have the same orientation then there is a t-permutation of D.

Proof. Note that by Claim 4 the neighbours of v_1 and v_2 have degree at least 3. Then $D'=D-\{v_1,v_2,u,x_1\}$, $\sigma=(x_1,v_1)(u,v_2)\sigma'$ or $\sigma=(u,v_1)(x_1,v_2)\sigma'$ where σ' is a t-permutation of D'. □

Definition 1. Let $k\geqslant 3$, $l\geqslant 0$. A *water-wheel* $W=W(k,l)$ with a center w is an induced subdigraph of D such that $V(W)=\{w,v_1,\ldots,v_k,x_1,\ldots,x_k,y_1,\ldots,y_l\}$, $x_i\neq w$, $i=1,\ldots,k$, $N(w,D)=\{v_1,\ldots v_k,y_1,\ldots,y_l\}$, $d(v_i,D)=2$, $N(v_i,D)=\{w,x_i\}$, $i=1,\ldots,k$, $d(y_j,D)\geqslant 3$, $j=1,\ldots,l$.

By Claims 4 and 5 it may be supposed that in every water-wheel $x_i\neq x_j$ for $i\neq j$, $d(x_i,D)\geqslant 3$ for $i=1,\ldots,k$ and $\{x_1,\ldots,x_k\}\cap\{v_1,\ldots,v_k\}=\emptyset$.

Claim 9. *If there is a water-wheel $W(k,l)$ and three vertices u_1,u_2,u_3 of degree 2 such that $N(v_i,D)\cap N(v_j,D)=\emptyset$ for $i\neq j$, $i,j=1,2,3$ and $N(v_p,D)\cap N(u_q,D)=\emptyset$ for p, $q=1,2,3$ (see the Figure 12) then there is a t-permutation of D.*

Proof. We use the notation introduced in Fig. 12. Let $A=\{v_1,v_2,v_3,u_1,u_2,u_3\}$, $D'=D-A$ and σ' be a t-permutation of D'. It is sufficient to consider only the case when $\sigma'[N(A,D)]=N(A,D)$ unless n is odd, one of the vertices of $N(A,D)$ is the fixed

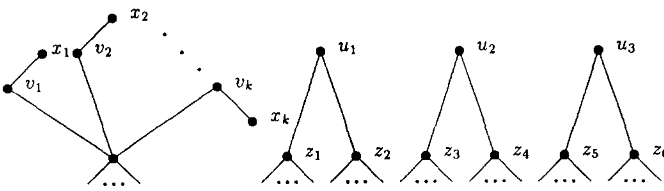

Fig. 12.

point of the permutation σ' and there is exactly one vertex a of $N(A,D)$ such that, $\sigma'(a) \notin \sigma'[N(A,D)]$. We consider three cases:

Case 1: $\sigma'(w) = x_1$.

Without loss of generality we may suppose that $\sigma'(x_2) \notin \{z_1,z_2\}$ and $\sigma'(x_3) \in \{z_3,z_4\}$. Then $\sigma = (u_1,v_2)(u_2,v_3)(v_1,u_3)\sigma'$.

Case 2: $\sigma'(w) = z_1$.

Observe that we always have one of the following three subcases: either $\sigma'(z_2) \in \{x_1,x_2,x_3\}$, $\sigma'(z_2) = u_1$ say or $\sigma'(z_2) \in \{z_3,z_4,z_5,z_6\}$, $\sigma'(z_2) = z_3$ or else z_2 is the fixed point of the permutation σ'.

Subcase 1: $\sigma'(z_2) = x_1$.

If $\sigma'(x_2) = x_3$ then $\sigma = (u_1,u_3)(v_1,v_2)(u_2,v_3)\sigma'$.

If $\sigma'(x_2) \neq x_3$ then $\sigma = (v_2,v_3)(v_1,u_2)(u_1,u_3)\sigma'$.

Subcase 2: $\sigma'(z_2) = z_3$.

Suppose first that there are $i,j \in \{1,2,3\}$, $i<j$, such that $\sigma'(x_i)=x_j$. Let $\sigma'(x_1)=x_2$. Then $\sigma = (u_1,u_3)(v_1,v_3)(v_2,u_2)\sigma'$. So we may assume that none of transpositions (x_1,x_2), (x_2,x_3) or (x_1,x_3) is the factor of the t-permutation σ'. Hence we may suppose that $\sigma'(x_1) = z_4$, or $\sigma'(x_1) = z_5$. If $\sigma'(x_1) = z_4$, then $\sigma = (v_1,v_3)(v_2,u_2)(u_1,u_3)\sigma'$, if $\sigma'(x_1) = z_5$ then $\sigma = (v_1,u_2)(v_2,v_3)(u_1,u_3)\sigma'$.

Subcase 3: $\sigma'(z_2) = z_2$.

We proceed as in Subcase 1 unless $\sigma'(x_2) \neq x_3$ and $\sigma(x_1) \in \{z_3,z_4\}$. In the last case $\sigma = (v_1,u_3)(u_1,u_2)(v_2,v_3)\sigma'$.

Case 3: $\sigma'(w) = w$.

At least one of the permutations $(v_1,u_i)(v_2,u_j)(v_3,u_k)\sigma'$ $(i \neq j \neq k \neq i)$ is a t-permutation of D. □

Claim 10. *If there is a water-wheel $W(k,0), k \geqslant 3$ in D then there is a t-permutation of D.*

Proof. Let $W = W(k,0)$ be a water-wheel, $V(W) = \{w, v_1, \ldots, v_k, x_1, \ldots, x_k\}$ (Fig. 13). If there are two vertices v_i and v_j $(i \neq j)$ such that the arcs $<v_i,x_i>$ and $<v_j,w>$ have the same orientation, then the proof is completed by Claim 8. Therefore, all the arcs $<x_i,v_i>$ and $<v_j,w>$ $(i,j=1,\ldots,k)$ have the same orientation. Without loss of generality we may suppose that $(x_i,v_i) \in A$ and $(v_i,w) \in A$ for $i=1,\ldots,k$. If n is odd then the proof follows by Claim 2. So we assume that n is even. By Claim 7 we may

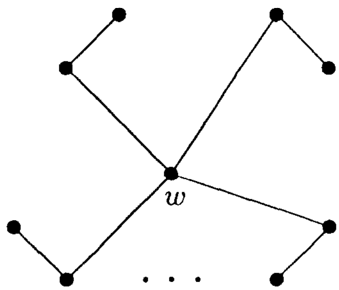

Fig. 13.

suppose that $k \geqslant 4$. Then $D' = D - \{v_1, v_2, v_3, v_4\}$ and the reader may easily check that one of the permutations, $(v_1, v_p)(v_q, v_r)\sigma'$ where $p \neq q \neq r \neq p$, is a t-permutation of D. □

Claim 11. *If there are at least two vertices v_1, v_2 such that $d(v_i,D)=2$ and $|N(v_i,D)|=1$ for $i=1,2$ then there is a t-permutation of D.*

Proof. By Claim 2 we may suppose that n is even and by Claim 3 that the neighbour of v_i has a degree greater than or equal to 3.

Case 1: There are two vertices v_1 and v_2 such that $d(v_1,D)=d(v_2,D)=2$, $|N(v_1,D)|=|N(v_2,D)|=1$ and $N(v_1,D)=N(v_2,D)$.

The digraph D is clearly embeddable in the required way ($D' = D - \{v_1, v_2\}$, $\sigma = (v_1, v_2)\sigma'$).

Case 2: There are three vertices v_1, v_2 and v_3 such that $d(v_i,D)=2$, $|N(v_i,D)|=1$ and $N(v_i,D)\cap N(v_j,D)=\emptyset$ for $i,j=1,2,3$ and $i \neq j$.

Let u_i be the neighbour of v_i, $i=1,2,3$ and $x \in V(D)-\{v_1,v_2,v_3,u_1,u_2,u_3\}$. Observe that $d(u_i,D) \geqslant 3$ for $i=1,2,3$. Set $D' = D - \{v_1,v_2,v_3,u_1,u_2,u\}$, where $u=u_3$ if there are at least nine arcs incident with $\{v_1,v_2,v_3,u_1,u_2,u_3\}$ or else $u=x$. Then $\sigma=(u_1,v_2)(u_2,v_3)(u,v_1)\sigma'$.

Case 3: There are three vertices v_1, v_2 and v_3 such that $d(v_iD)=2$ for $i=1,2,3$, $|N(v_i,D)|=1$ for $i=1,2$, $|N(v_3,D)|=2$, $N(v_1,D)\cap N(v_j,D)=\emptyset$ for $j=2,3$ and the neighbour of v_2 is adjacent with v_3.

We use the same arguments as in Case 2.

Case 4: There are four vertices v_1,v_2,v_3,v_4 of degree 2 with pairwise different neighbours such that $|N(v_i,D)|=1$ for $i=1,2$ and $|N(v_i,D)|=2$ for $i=3,4$.

Denote $N(v_i,D)=\{x_i\}$ for $i=1,2$ and $N(v_3,D)=\{x_3,x_4\}$, $N(v_4,D)=\{x_5,x_6\}$. Define $D'=D-\{v_1,v_2,v_3,v_4\}$. Observe that it is sufficient to consider σ', which is a t-permutation of D', such that $\sigma'(\{x_1,\ldots,x_6\})=\{x_1,\ldots,x_6\}$. If $\sigma'(x_1)=x_2$ then $\sigma=(v_1,v_3)(v_2,v_4)\sigma'$ is a t-permutation of D. Otherwise without loss of generality we assume that $\sigma'(x_1)=x_3$ and we consider two cases, $\sigma'(x_2)=x_4$ or $\sigma'(x_2)=x_5$. In the first one $\sigma=(v_1,v_2)(v_3,v_4)\sigma'$, in the second one $\sigma=(v_1,v_4)(v_2,v_3)\sigma'$ is a t-permutation of D.

Estimating the size of D we easily obtain:

Remark. Suppose that there is a water-wheel in D. Then there are at least four vertices of degree 2 not contained in this water-wheel.

By Claim 6 and the above remark we conclude that the cases 1–4 cover all possibilities with at least two vertices v_i of degree 2 such that $|N(v_i, D)| = 1$. $\square$

Claim 12. *If there is exactly one vertex v of degree 2 such that $|N(v,D)| = 1$. Then there is either a t-permutation of D or the neighbour u of v has degree at least 4 and all vertices of degree 2 adjacent with u have no other common neighbours with vertices of degree 2.*

Proof. By Claims 2 and 3 we may suppose that n is even and u has degree greater than or equal to 3. We consider two cases:

Case 1: There are two vertices v_1 and v_2 of degree 2 such that $N(v_1, D) = \{u, x_1\}$ and $N(v_2, D) = \{x_1, x_2\}$.

Then $D' = D - \{v, v_1, v_2, u, x_1, x_2\}$ and $\sigma = (v, x_2)(v_1, u)(x_1, v_2)\sigma'$ is a t-permutation of D.

Case 2: There are two vertices v_1 and v_2 of degree 2 with pairwise different neighbours such that $<u, v_i> \notin A$, $i = 1, 2$ and u has two neighbours.

Let $N(v_1, D) = \{x_1, x_2\}$, $N(v_2, D) = \{x_3, x_4\}$ and $N(u, D) = \{v, s\}$. Set $D' = D - \{v, v_1, v_2, u\}$ and σ' is a t-permutation of D'. Let us first suppose that $s \in \{x_1, \ldots, x_4\}$. For instance $s = x_1$. If $\sigma'(s) = x_2$ then $\sigma = (v, v_1)(u, v_2)\sigma'$. If, $\sigma'(s) \in \{x_3, x_4\}$, then $\sigma = (v, v_2)(u, v_1)\sigma'$. If $\sigma'(s) \notin \{x_2, x_3, x_4\}$ then σ is one of the two permutations described above. In every case σ is a t-permutation of D.

Let $s \notin \{x_1, \ldots, x_4\}$. If $\sigma'(s) \in \{x_1, \ldots, x_4\}$ and for example $\sigma'(s) = x_1$ or $\sigma'(s) \notin \{x_1, \ldots, x_4\}$ then $\sigma = (v, v_1)(u, v_2)\sigma'$ is a t-permutation of D. Observe that if u has two neighbours then there are always two vertices v_1, v_2 of degree 2 described in Case 2. $\square$

Claim 13. *Let every vertex of degree 2 has two different neighbours. If there exist exactly r water-wheels in the digraph $D, r = 0, 1, 2, 3$ then there is a t-permutation of D.*

Proof. The main idea of the proof is to estimate the size of D and then use Claims 5 and 9. We start with notation. Let W_i denote a water-wheel with center w_i for $i = 1, \ldots, r$. By t_w we denote the number of arcs joining two centers. A symmetric arc is calculated as one arc. We divide the centers' neighbours with at least degree three into r disjoint subsets. Let t_i denote the number of such vertices adjacent with exactly i centers, $i = 1, \ldots, r$. Now we divide the centers' neighbours with degree 2 into two subsets. Let p_j denote the number of such vertices adjacent with exactly j centers, $j = 1, 2$. A number c_k is the cardinality of the set C_k of vertices which are the common neighbours of two vertices of degree two of k different water-wheels. We ignore the centers in this calculation. Observe that in this case $k = 2$. By the Claim 7 the vertices of C_2 have a degree at least 4.

The path $P = (u_0, x_1, x_2, \ldots, x_{2k+1}, v_0)$ is a *wave* if $d(u_0, D) = d(v_0, D) = d(x_{2i}) = 2$ $(i = 1, \ldots, k)$ and the vertices u_0, v_0 and x_{2i} are not contained in any water-wheel. A wave P is *maximal* if it is not contained in a longer wave and u_0, v_0 are called the *end-vertices* of maximal wave P. Then f_i^{2l} is the number of maximal waves of length $2(l-1)$ such that end-vertices have i common neighbours with vertices of degree two in water-wheels (observe that $i \in \{0, 1, 2\}$ and the common neighbours have degree at least 4). If u is a vertex of degree 2 such that u has no common neighbours with any other vertices of degree two then it is a wave of length 0.

If there exists a wave of length at least 20, then there are at least six vertices of degree 2 such that any two of them have no common neighbours and there is a t-permutation of D by Claim 5. Observe that

$$\tfrac{1}{2}(t_1 + 2t_2 + 3t_3 + p_1 + p_2) \leqslant \sum_{w_i} d(w_i, D),$$

$$p_1 + p_2 \leqslant \sum_{v} d(v, D),$$

where v is a neighbour of w_i such that $d(v, D) = 2$.

$$\tfrac{3}{2}(t_1 + t_2 + t_3 - t_w) \leqslant \sum_{y} d(y, D),$$

where $y \neq w_i$, y is a neighbour of w_i such that $d(y, D) \geqslant 3$.

Estimating the size of D we obtain

$$\begin{aligned}
\tfrac{3}{2}(n-2) &> e(D) \\
&\geqslant \tfrac{1}{2}[t_1 + 2t_2 + 3t_3 + p_1 + 2p_2 + 2(p_1 + p_2) + 3(t_1 + t_2 + t_3 - t_w) + 4c_2 \\
&\quad + \sum_{i=1}^{10} f_0^{2l}(2l + 4(l-1)) + \sum_{i=1}^{10} f_1^{2l}(2l + 4l) + \sum_{i=1}^{10} f_2^{2l}(2l + 4(l+1)) \\
&\quad + 3(n - [r + p_1 + p_2 + t_1 + t_2 + t_3 - t_w + c_2 \\
&\quad + \sum_{i=1}^{10} f_0^{2l}(2l-1) + \sum_{i=1}^{10} f_1^{2l} 2l + \sum_{i=1}^{10} f_2^{2l}(2l+1)])].
\end{aligned}$$

Since $\sum_{i=1}^{10} f_0^{2l} > t_1 + 2t_2 + 3t_3 + p_2 + c_2 + \sum_{i=1}^{10} f_2^{2l} - 3r + 6$, by Claim 10, $t_1 + 2t_2 + 3t_3 \geqslant r$ and the Claim 6 implies that $q = p_2 + c_2 + \sum_{i=1}^{10} f_2^{2l} \leqslant \max\{0, r-1\}$. So there are r water-wheels and at least $q + 7 - 2r$ vertices of degree 2 with pairwise different neighbours in D. We can apply Claim 9 for $r = 1, 2, 3$ and Claim 5 for $r = 0$. □

Claim 14. *Let $r \in \{0, 1, 2, 3\}$. Suppose that there are exactly r vertices u_i, $i = 1, \ldots, r$ such that each of them is the common neighbour of at least three vertices of degree 2 with two different neighbours. Then there is a t-permutation of D.*

Proof. We use the same method as in the proof of Claim 13. By Claims 11 and 13 we may assume that there is exactly one vertex v of degree 2 such that $|N(v,D)| = 1$. Denote by u the neighbour of v. By Claim 12 we obtain four cases.

Case 1: The vertex v is the only neighbour of degree 2 of the vertex u and $d(u,D) \geqslant 4$. Observe that there are r water-wheels in D. In this case we repeat the proof of Claim 13 but with the following estimation of the size of D

$$\begin{aligned}
&\tfrac{3}{2}(n-2) > e(D)\\
&\geqslant \tfrac{1}{2}[t_1 + 2t_2 + 3t_3 + p_1 + 2p_2 + 2(p_1 + p_2) + 3(t_1 + t_2 + t_3 - t_w) + 4c_2\\
&+\sum_{i=1}^{10} f_0^{2l}(2l + 4(l-1)) + \sum_{i=1}^{10} f_1^{2l}(2l + 4l) + \sum_{i=1}^{10} f_2^{2l}(2l + 4(l+1))\\
&+3(n - [r + p_1 + p_2 + t_1 + t_2 + t_3 - t_w + c_2\\
&+\sum_{i=1}^{10} f_0^{2l}(2l-1) + \sum_{i=1}^{10} f_1^{2l}2l + \sum_{i=1}^{10} f_2^{2l}(2l+1)] - 2) + 2 + 4].
\end{aligned}$$

Case 2: The vertex u is the neighbour of an end-vertex of a wave f of length 0. Observe that then there are r water-wheels and the end-vertices of the wave f have no neighbours contained in any water-wheel. Estimating the size of D we obtain,

$$\begin{aligned}
&\tfrac{3}{2}(n-2) > e(D)\\
&\geqslant \tfrac{1}{2}[t_1 + 2t_2 + 3t_3 + p_1 + 2p_2 + 2(p_1 + p_2) + 3(t_1 + t_2 + t_3 - t_w) + 4c_2\\
&+\sum_{i=1}^{10} f_0^{2l}(2l + 4(l-1)) + \sum_{i=1}^{10} f_1^{2l}(2l + 4l) + \sum_{i=1}^{10} f_2^{2l}(2l + 4(l+1))\\
&+3(n - [r + p_1 + p_2 + t_1 + t_2 + t_3 - t_w + c_2\\
&+\sum_{i=1}^{10} f_0^{2l}(2l-1) + \sum_{i=1}^{10} f_1^{2l}2l + \sum_{i=1}^{10} f_2^{2l}(2l+1)] - 1) + 2].
\end{aligned}$$

Hence there are r water-wheels and at least $q+6-2r$ vertices of degree 2 with pairwise different neighbours such that their neighbours are not contained in any water-wheel. We can apply Claim 9 for $r = 1,2,3$ and Claim 5 for $r = 0$.

Case 3: The vertex u is the neighbour of exactly two vertices v_1, v_2 of degree 2 (then (v_1, u, v_2) is a wave of length 2).

Notice that there are r water-wheels in D and the end-vertices of f have no neighbours contained in any water-wheel. We estimate the size of D again and in this case we obtain

$$\begin{aligned}
&\tfrac{3}{2}(n-2) > e(D)\\
&\geqslant \tfrac{1}{2}[t_1 + 2t_2 + 3t_3 + p_1 + 2p_2 + 2(p_1 + p_2) + 3(t_1 + t_2 + t_3 - t_w) + 4c_2\\
&+\sum_{i=1}^{10} f_0^{2l}(2l + 4(l-1)) + \sum_{i=1}^{10} f_1^{2l}(2l + 4l) + \sum_{i=1}^{10} f_2^{2l}(2l + 4(l+1))
\end{aligned}$$

$$+3(n-[r+p_1+p_2+t_1+t_2+t_3-t_w+c_2$$
$$+\sum_{i=1}^{10} f_0^{2l}(2l-1)+\sum_{i=1}^{10} f_1^{2l}2l+\sum_{i=1}^{10} f_2^{2l}(2l+1)]-1)+2].$$

The rest of the proof runs as in Case 2.

Case 4: The vertex u is the neighbour of at least three vertices of degree 2 such that a subdigraph induced by their neighbourhood is a water-wheel. We apply the arguments of the proof of Claim 13, with w_1 replaced by u. □

Claim 15. *Let every vertex of degree 2 have two different neighbours. If there exist at least four water-wheels in the digraph D then, there is a t-permutation of D.*

Proof. Choose four water-wheels W_i with centers w_i $(i=1,\ldots,4)$. We use the same method as in the proof of Claim 13. We also follow the notation of this proof. Because of a greater number of water-wheels in D we need t_i for $i=1,\ldots,4$ and c_k for $k=2,3,4$. By Claim 7 the vertices of C_2 have the degree at least 4. By Claim 10 the vertices of C_l have degrees at least $l+1$, for $l=3,4$. Let m denote the number of vertices of degree 2 that are not contained in the four water-wheels.

Observe that

$$\tfrac{1}{2}(t_1+2t_2+3t_3+4t_4+p_1+p_2)\leqslant \sum_{w_i} d(w_i,D);$$

$$p_1+p_2\leqslant \sum_{v} d(v,D),$$

where v is a neighbour of w_i such that $d(v,D)=2$.

$$\tfrac{3}{2}(t_1+t_2+t_3+t_4-t_w)\leqslant \sum_{y} d(y,D),$$

where $y\neq w_i$, y is a neighbour of w_i such that $d(y,D)\geqslant 3$.

We may estimate the size of D

$$\begin{aligned}\tfrac{3}{2}(n-2) &> e(D)\\ &\geqslant \tfrac{1}{2}[t_1+2t_2+3t_3+4t_4+p_1+2p_2+2(p_1+p_2)+3(t_1+t_2+t_3+t_4-t_w)\\ &\quad +4c_2+4c_3+5c_4+2m\\ &\quad +3(n-[4+t_1+t_2+t_3+t_4-t_w+p_1+p_2+c_2+c_3+c_4+m])].\end{aligned}$$

By Claim 10, $t_1+2t_2+3t_3+4t_4\geqslant 4$. Since $m\geqslant -1+p_2+c_2+c_3+2c_4$ and Claim 6 implies that $0\leqslant c_4+c_3\leqslant 1$ and $0\leqslant p_2+c_2\leqslant 3$. Moreover, if $c_4=1$, then $c_3=c_2=p_2=0$ and if $c_3=1$, then $c_2+p_2\leqslant 1$. We consider seven cases:

1. $c_4=1$ $(m\geqslant 1)$;
2. $c_4=0$, $c_3=1$, $c_2+p_2=1$ $(m\geqslant 1)$;
3. $c_4=0$, $c_3=1$, $c_2+p_2=0$ $(m\geqslant 0)$;
4. $c_4=0$, $c_3=0$, $c_2+p_2=3$ $(m\geqslant 2)$;
5. $c_4=0$, $c_3=0$, $c_2+p_2=2$ $(m\geqslant 1)$;

6. $c_4 = 0,\ c_3 = 0,\ c_2 + p_2 = 1\ (m \geqslant 0)$;
7. $c_4 = 0,\ c_3 = 0,\ c_2 + p_2 = 0\ (m \geqslant 0)$;

In every case we may find a water-wheel and three vertices u_1, u_2, u_3 of degree 2 such that the vertices u_1, u_2, u_3 have no common neighbours with the vertices of degree 2 in the water-wheel and they have pairwise different neighbours. So we can apply Claim 9. $\square$

Claim 16. *Suppose that there are at least four vertices u_i such that each of them is the common neighbour of at least three vertices of degree 2 with two neighbours. Then there is a t-permutation of D.*

Proof. We use the same method as in the proof of the Claim 15. We follow the notation of this proof but now m denotes the number of the vertices of degree 2 which are not adjacent with any u_i and which have two different neighbours. By Claims 11 and 15 we may assume that there is only one vertex v of degree 2 such that $N(v, D) = \{u\}$. By Claim 12 we obtain the same four cases as in the proof of Claim 14. In the first one we can repeat the proof of Claim 15. In the second and third we obtain at least four water-wheels and at least one vertex of degree 2 with two neighbours not contained in any of these water-wheels. Then we may apply Claim 9. In the last case we repeat the proof of Claim 15, with w_1 replaced by u. $\square$

Notice that the proofs of Claims 14 and 16 end the proof of Lemma 4. $\square$

By estimating the size of D we obtain that $\delta(D) \leqslant 2$ which completes the proof of Theorem 4. $\square$

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References

[1] A. Benhocine and A.P. Wojda, On self-complementation, J. Graph Theory 8 (1985) 335–341.

[2] B. Bollobás and S.E. Eldridge, Packings of graphs and applications to computational complexity, J. Combin. Theory Ser. B 25 (1978) 105–124.

[3] D. Burns and S. Schuster, Every $(p, p-2)$ graph is contained in its complement, J. Graph Theory 1 (1977) 277–279.

[4] D. Burns and S. Schuster, Embedding $(p, p-1)$ graphs in their complements, Israel J. Math. 30 (1978) 313–320.

[5] R.J. Faudree, C.C. Rousseau, R.H. Schelp and S. Schuster, Embedding graphs in their complements, Czechoslovak Math. J. 31 (106) (1981) 53–62.

[6] B. Ganter, J. Pelikán and L. Teirlinck, Small sprawling systems of equicardinal sets, Ars Combin. 4 (1977) 133–142.

[7] N. Sauer and J. Spencer, Edge disjoint placement of graphs, J. Combin. Theory Ser. B 25 (1978) 295–302.
[8] M. Woźniak, Embedding graphs of small size, Discrete App. Math. 51 (1994) 233–241.
[9] H.P. Yap, Packing of graphs – a survey, Discrete Math. 72 (1988) 395–404.